A Related Rates Approach to Modeling Branch Growth

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Why Branch Growth?

• Existing branch/knot profile models are static “snapshot” descriptions
• Adequacy for describing the profile in trees after respond to a treatment ???
• 1-2 recent branch modeling authors suggest that growth models for branches should be investigated

“It started out as just a little knot on the floor.”
Branches become Knots in Products

Degrade performance of structural members

Degrade appearance and ability to reman large cuttings
Silviculture Changes Branchiness:
(SMC Type I: Longbell Road)

ISPA: 540
SPA:
smaller branches

ISPA/2: 270
SPA:
larger branches

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Branches Growth Responds to Stem Growth

- Thinned & fertilized 4 years ago
- Branch growth is related to the stem growth
Related Rates Problems

• A balloon is being filled with air. How fast does its surface area grow as it increases in diameter? When the diameter is 6 inches?

• A 10 foot tall conical tank that with a 4 foot top radius is being filled with water at 2 ft$^3$ per minute. How fast is the water level rising when the tank is half-full?

• Two months ago, an insect outbreak began and is spreading at the rate of 10 miles per week. How fast is the infested area growing now?

• At the 7th whorl from the top of a Douglas-fir tree, the stem is increasing in diameter by 1/4 inch per year. How fast are the branches in the whorl growing?
Related Rates

• To solve these
  1. write appropriate equations describing the situation
  2. obtain derivatives (rates of change with respect to time), using implicit differentiation
  3. Plug in known information and solve
  4.
• In many biological situations, the equation in Step 1 is an allometric equation
Allometry & Related Rates

- If X and Y are measurements of parts of an organism, they are often related by the allometric equation

\[ Y = aX^k \]

- \( a, k > 0 \)

- Both Y and X change with age of the organism (time)
Allometry & Related Rates

• The growth rates of Y and X with respect to time are

\[
\frac{dY}{dt} \quad \text{and} \quad \frac{dX}{dt}
\]

• The relation between these growth rates can be obtained directly by applying implicit differentiation to the allometric equation
Allometry & Related Rates

\[ \frac{dY}{dt} = a k X^{k-1} \frac{dX}{dt} \]

Units of Y and X may not be compatible so divide both sides by

\[ Y = a X^k \]

and do some algebraic simplification to get

\[ \left( \frac{1}{Y} \frac{dY}{dt} \right) = k \left( \frac{1}{X} \frac{dX}{dt} \right) \]

% growth rate of Y  % growth rate of X

“specific” growth rate of Y  “specific” growth rate of X
Allometry & Related Rates

\[
\left\{ \left( \frac{1}{Y} \frac{dY}{dt} \right) \div \left( \frac{1}{X} \frac{dX}{dt} \right) \right\} = k
\]

- The percentage (or specific) rates of growth are directly proportional

- For many varieties of fish the length vs weight relationship is approximately

\[\text{Weight} = a \text{Length}^3\]

- a 10% increase in length translates into a 30% increase in weight
Re-casting in terms of branch & stem growth of a tree

• Replace Y by branch diameter (BD) and X by stem diameter (SD)
  – \( \frac{dBD}{dt} = \) branch diameter growth rate
  – \( \frac{dSD}{dt} = \) stem diameter growth rate (might know this from a growth model)

\[
\left\{ \left( \frac{1}{BD} \frac{dBD}{dt} \right) \bigg/ \left( \frac{1}{SD} \frac{dSD}{dt} \right) \right\} = k
\]
Questions

\[
\left\{ \frac{1}{BD} \frac{dB_D}{dt} \right\} / \left( \frac{1}{SD} \frac{dS_D}{dt} \right) = k
\]

- Is $k$ really constant for all conditions?
- If $k$ is not constant, does it change systematically with competition, treatments, etc. so we can model its behavior?
• Possible trajectories of “k” for a single branch

• Open grown tree
  – Build 3-5 years’ retained needles
    • Relative branch growth rate may increase
  – Then in needle balance stage; shed old needles & add new
    • Branch is less efficient
      – Shade from higher branches
    • Longer, larger branch
      – consumes more photosynthate
      – goes into deficit
      – may parasitize bole to stay alive

$$\left\{ \frac{1}{BD} \frac{dBD}{dt} \right\} / \left( \frac{1}{SD} \frac{dSD}{dt} \right) = k$$
• Possible trajectories of “k” for a single branch
  • Tree in dense stand
  • Inter tree competition
    • Larger branch consumes more of photosynthate
    • Branch is less efficient
      – Shade from higher branches
      – Shading from competing trees
      – Crown recession and branch death; k = 0

\[
\left\{ \frac{1}{BD} \frac{dBD}{dt} \div \left( \frac{1}{SD} \frac{dSD}{dt} \right) \right\} = k
\]

Branch age, depth in crown
• Possible trajectories of “k” for a single branch
  • Thinning
    – Reduces inter-tree competition & shading;
      • Build denser foliage
      • Increase branch growth
      • But, trees grow faster and shading competition soon returns

\[
\left\{ \frac{1}{BD} \frac{dBD}{dt} \right\} \div \left( \frac{1}{SD} \frac{dSD}{dt} \right) = k
\]
• Possible trajectories of “k” for a single branch
• Fertilization
  – Adds growth resources to site
    • Build denser foliage
    • Increase branch growth
    • Trees grow faster and shading competition intensifies
      accelerating crown recession and branch death

\[
\left\{ \frac{1}{BD} \frac{dB}{dt} \right\} / \left\{ \frac{1}{SD} \frac{dSD}{dt} \right\} = k
\]
Sample Installations

• # 713 (Sauk Mountain)
  • Sedro Wooley, WA.
  • BH age = 4 in 1988
  • Site Index 120
  • Originally ~550 spa
    – Plot 1 (ISPA)
    – Plot 12 (ISPA + 200F)
    – Plot 6 (ISPA/2)
    – Plot 8 (ISPA/2 +BTS)
    – Plot 2 (ISPA/4)
    – Plot 11 (ISPA/4 +200F)

• # 736 (“Twin Peaks”)
  • Snoqualmie, Wa.
  • Planted 1984 with 2-0’s
  • Site Index 120
  • Originally ~ 460 spa
    – Plot 3 (ISPA)
    – Plot 7 (ISPA + 200F)
    – Plot 2 (ISPA/2)
    – Plot 11 (ISPA/2 +BTS)
    – Plot 1 (ISPA/4)
    – Plot 9 (ISPA/4 +200F)
Sample Trees

• 4 trees per plot
  – QMD
  – the 24\textsuperscript{th}, 63\textsuperscript{rd}, and 93\textsuperscript{rd} percentiles of the DBH distribution

• 2 Inst x 6 plots x 4 trees = 48 total trees
Sample Tree Measurements and Disks

• At every 3rd whorl from top
  – Disk from stem
  – Disk from and largest branch at
• Measure
  – dbh
  – total height
  – height to base of live crown
  – height to whorl and to disk; can calculate height growth
• Also have plot (stand)
  summary statistics
Stem & Largest Branch Disk from Whorl 12

- Scan and measure ring width along 4 perpendicular radii from pith
- On branch radii are vertical & horizontal directions with respect to the stem
- White strip = 4 inches
- Scanning is done on green samples
Branch & Stem % Growth at Whorls 9 and 12 of a Single Tree
"K" value trajectories

whorl 12

whorl 9
Status

- Tree sampling has been completed
- Most of the scanning data has been collected
- Analysis will be done May-June
- Paper presented at IUFRO in September